EE 345S Real-Time Digital Signal Processing Lab

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All-Pass Filters

- An all-pass filter has a magnitude response that is constant for all frequencies. The phase response may or may not be linear.
- A simple all-pass filter is the gain, i.e. $y(t) = g\,x(t)$ or $y[n] = g\,x[n]$ where x is the input and y is the output. The impulse response is $h(t) = g\,\delta(t)$ or $h[n] = g\,\delta[n]$. The frequency response is simply equal to g.
- Another simple all-pass filter is the ideal delay, i.e. $y(t) = x(t - t_0)$ or $y[n] = x[n - n_0]$ where t_0 and n_0 are constants. Impulse response is $h(t) = \delta(t - t_0)$ or $h[n] = \delta[n - n_0]$. The frequency response is $H_{freq}(f) = e^{-j 2 \pi f t_0}$ or $H_{freq}(\omega) = e^{-j \omega n_0}$. The magnitude response is equal to one in either case. Phase response is linear.
- A cascade of a gain and an ideal delay also has an all-pass response.
- A first-order IIR filter with one real-valued pole and one real-valued zero is all-pass if the zero location is equal to the reciprocal of the pole location:

$$H(z) = \frac{z - \frac{1}{r}}{z - r} \longrightarrow H_{freq}(\omega) = \frac{e^{j\omega} - \frac{1}{r}}{e^{j\omega} - r}$$

assuming that $|r| < 1$ for asymptotic stability. Magnitude response is

$$|H_{freq}(\omega)| = \left| \frac{e^{j\omega} - \frac{1}{r}}{e^{j\omega} - r} \right| = \frac{\left| e^{j\omega} - \frac{1}{r} \right|}{\left| e^{j\omega} - r \right|}$$

Here, $|e^{j\omega} - a| = |(\cos\omega - a) + j \sin\omega| = \sqrt{(\cos\omega - a)^2 + \sin^2\omega}$.

$$= \sqrt{\cos^2\omega - 2a\cos\omega + a^2 + \sin^2\omega} = \sqrt{a^2 - 2a\cos\omega + 1}$$

$$|H_{freq}(\omega)| = \frac{\sqrt{\frac{1}{r^2} - \frac{2}{r}\cos\omega + 1}}{\sqrt{r^2 - 2r\cos\omega + 1}} = \frac{\sqrt{\frac{1}{r^2}\left(r^2 - 2r\cos\omega + 1\right)}}{\sqrt{r^2 - 2r\cos\omega + 1}} = \frac{1}{|r|}$$

- A first-order IIR filter with one complex-valued pole and one complex-valued zero is all-pass if the zero radius is the reciprocal to the pole radius and if the angles are the same:

$$H(z) = \frac{z - \frac{1}{r_o} e^{j\omega_o}}{z - r_o e^{j\omega_o}} \Rightarrow H_{freq}(\omega) = \frac{e^{j\omega} - \frac{1}{r_o} e^{j\omega_o}}{e^{j\omega} - r_o e^{j\omega_o}}$$

assuming that $r_o < 1$ for asymptotic stability. The magnitude response is

$$|H_{freq}(\omega)| = \frac{|e^{j\omega} - \frac{1}{r_o} e^{j\omega_o}|}{|e^{j\omega} - r_o e^{j\omega_o}|}$$

Here,
$$|e^{j\omega} - (a+jb)| = |(\cos\omega - a) + j(\sin\omega - b)|$$
$$= \sqrt{(\cos\omega - a)^2 + (\sin\omega - b)^2}$$
$$= \sqrt{\cos^2\omega - 2a\cos\omega + a^2 + \sin^2\omega - 2b\sin\omega + b^2}$$
$$= \sqrt{(a^2+b^2) - 2\sqrt{a^2+b^2}\cos(\omega+\theta) + 1}$$

where $\theta = \arctan\left(-\frac{b}{a}\right)$.

$$|H_{freq}(\omega)| = \frac{\sqrt{\frac{1}{r_o^2} - \frac{2}{r_o}\cos(\omega+\theta) + 1}}{\sqrt{r_o^2 - 2r_o\cos(\omega+\phi) + 1}}$$

where $\theta = \arctan\left(-\frac{\frac{1}{r_o}\sin\omega_o}{\frac{1}{r_o}\cos\omega_o}\right) = -\omega_o$

$$\phi = \arctan\left(-\frac{r_o \sin\omega_o}{r_o\cos\omega_o}\right) = -\omega_o$$

Therefore,

$$|H_{freq}(\omega)| = \frac{\sqrt{\frac{1}{r_o^2} - \frac{2}{r_o}\cos(\omega-\omega_o) + 1}}{\sqrt{r_o^2 - 2r_o\cos(\omega-\omega_o) + 1}}$$

$$= \frac{\sqrt{\frac{1}{r_o^2}(r_o^2 - 2r_o\cos(\omega-\omega_o) + 1)}}{\sqrt{r_o^2 - 2r_o\cos(\omega-\omega_o) + 1}} = \frac{1}{r_o}$$